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Linear Algebra and its Applications 299 (1999) 175–190

LINEAR ALGEBRA AND ITS APPLICATIONS

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Hook immanantal inequalities for Hadamard's function

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Received 27 July 1999; accepted 12 August 1999

Submitted by R.A. Brualdi

Abstract

For an $n \times n$ positive semi-definite (psd) matrix A, Peter Heyfron showed in [9] that the normalized hook immanants, $\bar{d}_k$, $k = 1, \ldots, n$, satisfy the dominance ordering

$$\operatorname{per}(A) = \bar{d}_n(A) \geqslant \bar{d}_{n-1}(A) \geqslant \cdots \geqslant \bar{d}_2(A) \geqslant \bar{d}_1(A) = \det(A). \tag{a}$$

The classical Hadamard–Marcus inequalities assert that for an $n \times n$ psd matrix $A = [a_{ij}]$,

$$\operatorname{per}(A) = \bar{d}_n(A) \geqslant \prod_{i=1}^{n} a_{ii} \geqslant \bar{d}_1(A) = \det(A). \tag{b}$$

In view of the Hadamard–Marcus inequalities, it is natural to ask where the term $\prod_{i=1}^{n} a_{ii}$ sits in the family of descending normalized hook immanants in (a). More specifically, for each $n \times n$ psd A one wishes to determine the smallest $\kappa(A)$ such that

$$\bar{d}_{\kappa(A)}(A) \geqslant \prod_{i=1}^{n} a_{ii} \geqslant \bar{d}_{\kappa(A)-1}(A). \tag{c}$$

Heyfron [10] (see also [11,17]) established for all $n \times n$ psd A that $\kappa(A) \geqslant \min\{n-2, 1+\sqrt{n-1}\}$.

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In this work, we focus on the case where A is the Laplacian matrix of a tree T. It is meaningful to seek bounds on $\kappa(A)$ that depend on some topological features of the tree T such as the size of a maximum matching in T. For a tree T on $n \geqslant 2$ vertices with a maximum matching of size m, we show that $\lceil n/2 + m/3 \rceil \geqslant \kappa(A) \geqslant \lceil (n+1)/2 \rceil$. Both these bounds on $\kappa(A)$ are tight and the coefficient $1/3$ for the term in m in the upper bound cannot be lowered to $1/4$.

AMS classification: 05C05; 15A15

Keywords: Hadamard's function; Hook characters; Immanants

1. Introduction

Let H_n denote the collection of $n \times n$ positive semi-definite (psd) matrices. For a matrix $A = [a_{ij}]$, the Hadamard function $h(\cdot)$ is defined as $h(A) := \prod_{i=1}^{n} a_{ii}$, which is the product of the diagonal entries of A.

The classical inequalities of M. Marcus and Jacques Hadamard state that for all $A \in H_n$,

$$\operatorname{per}(A) \geqslant h(A) \geqslant \det(A), \tag{1}$$

where $\operatorname{per}(A)$ and $\det(A)$ are the permanent and the determinant of A, respectively. The upper and lower bounds are due to Marcus [13] and Hadamard [8], respectively. Since the permanent and determinant are examples of normalized immanants, it is natural to explore such inequalities for other normalized immanants. An immanant d_λ is a matrix function that is associated with an irreducible character χ_λ of the symmetric group S_n, indexed by the partition λ of n. For an $n \times n$ matrix $A = [a_{ij}]$, the immanant $d_\lambda(A)$ is defined by

$$d_\lambda(A) := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

The normalized immanant $\bar{d}_\lambda$ is obtained by setting $\bar{d}_\lambda(A) := (d_\lambda(A))/(\chi_\lambda(id))$ where *id* denotes the identity permutation of S_n. The permanent and the determinant functions correspond to the trivial character $\chi_{(n)}$, and the alternating character $\chi_{(1^n)}$, respectively. That is

$$\operatorname{per}(A) = \bar{d}_{(n)}(A) = d_{(n)}(A) \quad \text{and} \quad \det(A) = \bar{d}_{(1^n)}(A) = d_{(1^n)}(A).$$

The problem of characterizing the normalized immanants $\bar{d}_\lambda$ for which

$$\bar{d}_\lambda(A) \geqslant h(A)$$

for all $A \in H_n$ has been investigated by several authors in [7,8,10,11,13,17].

In connection with the permanental dominance conjecture that asserts for any partition λ of n,

$$\operatorname{per}(A) = \bar{d}_{(n)}(A) \geqslant \bar{d}_\lambda(A)$$

for all $A \in H_n$, Peter Heyfron [9] showed that the normalized hook immanants $\bar{d}_{(k,1^{n-k})}$ are ordered as

$$\begin{aligned}\operatorname{per}(A) &= \bar{d}_{(n)}(A) \geqslant \cdots \geqslant \bar{d}_{(k+1,1^{n-k-1})}(A)\\ &\geqslant \bar{d}_{(k,1^{n-k})}(A) \geqslant \cdots \geqslant \bar{d}_{(1^n)}(A) = \det(A)\end{aligned} \tag{2}$$

In view of (1), it is of interest to ask where the term $h(A)$ sits in the sequence of inequalities in (2).

A special case of a conjecture by Russell Merris [18], and also suggested in [12], asserts that if $n > 2k$ then

$$h(A) \geqslant \bar{d}_{(k,1^{n-k})}(A) \tag{3}$$

for all $A \in H_n$. The inequality in (3) has been established for some small values of k in [10,18] but it is not true in general as shown in Proposition 2 in [11] by Gordon James and Heyfron.

When we restrict the study of these inequalities to a smaller family of psd matrices such as the Laplacian matrices of trees, we can generally expect sharper results. It turns out that the conjectured inequality in (3) holds when A is the Laplacian matrix of a tree. For a tree T on $n \geqslant 2$ vertices with a maximum matching of size m and Laplacian matrix $L(T)$, we wish to determine the index $\kappa(T) \geqslant 2$ for which

$$\bar{d}_{(\kappa(T),1^{n-\kappa(T)})}(L(T)) \geqslant h(L(T)) > \bar{d}_{(\kappa(T)-1,1^{n-\kappa(T)+1})}(L(T)).$$

Using a recently derived expression for the hook immanants of tree Laplacians we show that

$$\left\lceil \frac{n}{2} + \frac{m}{3} \right\rceil \geqslant \kappa(T) \geqslant \left\lceil \frac{n+1}{2} \right\rceil.$$

The upper and lower bounds on $\kappa(T)$ are tight. More generally, one can seek the best coefficients κ_1 and κ_2 for which

$$\left\lceil \frac{n}{2} + \kappa_1 m \right\rceil \geqslant \kappa(T) \geqslant \left\lceil \frac{n}{2} + \kappa_2 m \right\rceil. \tag{4}$$

We establish that $1/3 \geqslant \kappa_1 > 1/4$.

2. Vertex orientations and hook immanants

Let T be a tree with n vertices $v_1, v_2, \ldots, v_n$ and edge set $E(T)$. For terms concerning graph theory, we refer the reader to [1]. The Laplacian matrix $L(T) = [l_{ij}]$ of the tree T is an $n \times n$ matrix defined by

$$l_{ij} := \begin{cases} \deg_T(v_i) & \text{if } i = j,\\ -1 & \text{if } \{v_i, v_j\} \in E(T),\\ 0 & \text{otherwise,}\end{cases}$$

where $\deg_T(v_i)$ is the degree of the vertex v_i in the tree T.

A vertex orientation of a tree is achieved by assigning for each vertex in T an arrow, pointing away from the vertex along one of its incident edges. It should be noted that some edges may have no arrows on them and some others will have two arrows on them. An orientation of vertices where exactly j edges have two arrows on them will be called a j-vertex orientation. In a j-vertex orientation, the edges with two arrows on them form a matching of size j in the tree. For each $j = 0, 1, \ldots, \lfloor n/2 \rfloor$, let $a_T(j)$ denote the total number of j-vertex orientations for the tree T.

In this paper, we let m denote the largest possible number of edges having two arrows on them among all the vertex orientations of the tree T. That is, for all $j > m$, $a_T(j) = 0$ and $a_T(m) \neq 0$. This number m is also the size of a maximum matching in T. It is observed in [3] that $a_T(0) = 0$ and $a_T(1) = n - 1$. Fig. 1 shows some examples of vertex orientations.

Immanants of Laplacian matrices of trees have been explored in several works [3–6,14,15]. The following formula that expresses the hook immanants as weighted combinations of vertex orientations may be found in [3].

For $k = 1, 2, \ldots, n$ we have

$$\bar{d}_{(k,1^{n-k})}(L(T)) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times 2^j \times \frac{\binom{n-j-1}{k-j-1}}{\binom{n-1}{k-1}}.$$

Simplifying, we get

$$\begin{aligned}
\bar{d}_{(k,1^{n-k})}(L(T)) &= \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times 2^j \times \frac{(n-j-1)!}{(n-k)!(k-j-1)!} \times \frac{(n-k)!(k-1)!}{(n-1)!} \\
&= \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times 2^j \times \frac{(k-1)(k-2)\cdots(k-j)}{(n-1)(n-2)\cdots(n-j)} \\
&= \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times \alpha(n, k; j),
\end{aligned}$$

where

$$\alpha(n, k; j) := 2^j \times \frac{(k-1)(k-2)\cdots(k-j)}{(n-1)(n-2)\cdots(n-j)}.$$

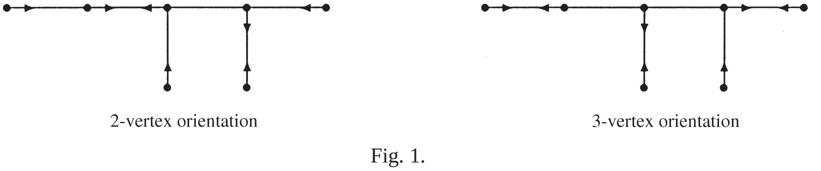

Fig. 1.

The next result suggests why vertex orientations may be a convenient tool in the study of the Hadamard function for Laplacian matrices of trees.

Lemma 2.1. *Let T be a tree on n vertices with Laplacian matrix $L(T) = [l_{ij}]$. Then $h(L(T)) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j)$.*

Proof. Consider all the possible vertex orientations of a tree T whose vertices are $v_1, v_2, \ldots, v_n$. Each vertex orientation falls into exactly one category, namely, 1-vertex orientation, 2-vertex orientation, ..., $\lfloor n/2 \rfloor$-vertex orientation. The total number of vertex orientations of T is thus equal to $\sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j)$.

On the other hand, for each vertex $v_i \in V(T)$, there are $\deg_T(v_i)$ ways of assigning the arrow to this vertex in a vertex orientation of T. Thus the total number of vertex orientations of T is $\prod_{i=1}^{n} \deg_T(v_i)$. It follows that

$$h(L(T)) = \prod_{i=1}^{n} l_{ii} = \prod_{i=1}^{n} \deg_T(v_i) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j). \qquad \square$$

For convenience, we shall denote $h(L(T)), \kappa(L(T))$ and $\bar{d}_{(k,1^{n-k})}(L(T))$ by $h(T), \kappa(T)$ and $\bar{d}_k(T)$, respectively. The following result proves that the conjectured inequality in (3) holds for Laplacian matrices of trees.

Theorem 2.2. *For all trees T on $n \geqslant 2$ vertices, $h(T) > \bar{d}_{\lfloor n/2 \rfloor}(T)$. That is, $\kappa(T) \geqslant \lceil (n+1)/2 \rceil$.*

Proof. For $j = 1, \ldots, \lfloor n/2 \rfloor$,

$$\begin{aligned}
\alpha\left(n, \left\lfloor \frac{n}{2} \right\rfloor; j\right) &= 2^j \times \frac{(\lfloor n/2 \rfloor - 1)(\lfloor n/2 \rfloor - 2) \cdots (\lfloor n/2 \rfloor - j)}{(n-1)(n-2) \cdots (n-j)} \\
&\leqslant \frac{(n-2)(n-4) \cdots (n-2j)}{(n-1)(n-2) \cdots (n-j)} \\
&< 1.
\end{aligned}$$

Now,

$$\bar{d}_{\lfloor n/2 \rfloor}(T) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times \alpha\left(n, \lfloor n/2 \rfloor; j\right) < \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) = h(T)$$

since $a_T(1) = n - 1 > 0$. It follows that $\kappa(T) \geqslant \lceil (n+1)/2 \rceil$. $\square$

This lower bound on $\kappa(T)$ is attained for the star s_n which is a tree on n vertices with one central vertex of degree $n-1$ and $n-1$ leaves. Here $h(s_n) = n - 1$.

From [2, Eq. (9)] we have $\bar{d}_k(s_n) = 2(k-1)$. The smallest k for which $\bar{d}_k(s_n) = 2(k-1) \geqslant n-1 = h(s_n)$ is $k = \lceil (n+1)/2 \rceil$. Thus $\kappa(s_n) = \lceil (n+1)/2 \rceil$.

We observe that

$$\begin{aligned}\bar{d}_k(T) - h(T) &= \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times \alpha(n,k;\, j) - \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j)\\ &= \sum_{j=1}^{\lfloor n/2 \rfloor} a_T(j) \times (\alpha(n,k;\, j) - 1)\\ &= \sum_{j=1}^{m} a_T(j) \times (\alpha(n,k;\, j) - 1),\end{aligned} \tag{5}$$

where m is the size of a maximum matching in T. If $k \geqslant 2$ is such that $\alpha(n,k;\, j) \geqslant 1$, for all $j = 1, \ldots, m$ then $\bar{d}_k(T) \geqslant h(T)$ and so $k \geqslant \kappa(T)$. This would be our main strategy in obtaining bounds on $\kappa(T)$. With a few exceptions, $\alpha(n, \lceil n/2 + m/3 \rceil;\, j) > 1$ for $j = 1, \ldots, m$. Handling these exceptional cases by other arguments, we show that the bound $\lceil n/2 + m/3 \rceil \geqslant \kappa(T)$ is valid in general. For the case of the star s_n we have $m = 1$ and $\lceil n/2 + 1/3 \rceil = \kappa(s_n) = \lceil (n+1)/2 \rceil$.

3. Some properties of $\alpha(n, \lceil n/2 + m/3 \rceil;\, j)$

We begin by documenting some properties of the numbers $\alpha(n, \lceil n/2 + m/3 \rceil;\, j)$ for $j = 1, \ldots, m$.

Lemma 3.1. *For fixed n and m where $1 \leqslant m \leqslant \lfloor n/2 \rfloor$, the sequence of numbers is*

$$\left\{\alpha\left(n, \left\lceil \frac{n}{2} + \frac{m}{3} \right\rceil;\, j\right) : j = 1, \ldots, m\right\}$$

unimodal.

Proof. Let $\gamma := \lceil n/2 + m/3 \rceil$ and $\tilde{\alpha}_j := \alpha(n, \lceil n/2 + m/3 \rceil;\, j)$ for $j = 1, \ldots, m$. We recall that

$$\tilde{\alpha}_j = 2^j \times \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-j)}{(n-1)(n-2)\cdots(n-j)} = \prod_{i=1}^{j} \frac{2(\gamma-i)}{n-i}.$$

We have

$$\begin{aligned}\tilde{\alpha}_j < \tilde{\alpha}_{j+1} \iff & \prod_{i=1}^{j} \frac{2(\gamma-i)}{(n-i)} < \prod_{i=1}^{j+1} \frac{2(\gamma-i)}{(n-i)}\\ \iff & n - j - 1 < 2(\gamma - j - 1)\\ \iff & j < 2\gamma - n - 1.\end{aligned}$$

Now $\tilde{\alpha}_j > \tilde{\alpha}_{j+1}$ for $j > 2\gamma - n - 1$. It is easy to check that $\tilde{\alpha}_{2\gamma-n-1} = \tilde{\alpha}_{2\gamma-n}$. Thus

$$\tilde{\alpha}_1 < \tilde{\alpha}_2 < \cdots < \tilde{\alpha}_{2\gamma-n-2} < \tilde{\alpha}_{2\gamma-n-1} = \tilde{\alpha}_{2\gamma-n}$$

and

$$\tilde{\alpha}_{2\gamma-n-1} = \tilde{\alpha}_{2\gamma-n} > \tilde{\alpha}_{2\gamma-n+1} > \cdots > \tilde{\alpha}_m. \qquad \square$$

Corollary 3.2. *The minimum value of the sequence* $\{\tilde{\alpha}_j \colon j = 1, \ldots, m\}$ *occurs at either* $\tilde{\alpha}_1$ *or* $\tilde{\alpha}_m$. *Moreover for* $m \geqslant 2$, $\tilde{\alpha}_1 > 1$.

Proof. For $m \geqslant 2$,

$$\tilde{\alpha}_1 = \frac{2(\gamma - 1)}{n-1} = \frac{(2\lceil n/2 + m/3\rceil - 2)}{n-1} \geqslant \frac{(n + 2m/3 - 2)}{n-1} > 1. \qquad \square$$

This means that for all trees other than the star, $\tilde{\alpha}_1 > 1$. We next investigate when $\tilde{\alpha}_m \geqslant 1$. Let m be fixed and let $n \geqslant 2m$ vary. To remind ourselves that the quantities depend on m and n, we define $\gamma_{m,n} := \lceil n/2 + m/3\rceil$ and $\tilde{\alpha}_{m,n} := \alpha(n, \lceil n/2 + m/3\rceil; m)$.

Lemma 3.3. *Let* $m \geqslant 2$ *be an integer. Consider the following two types of sequences:*

$$A_m = \{\tilde{\alpha}_{m,2t} : t = m, m+1, \ldots\}$$

$$B_m = \{\tilde{\alpha}_{m,2t+1} : t = m, m+1, \ldots\}.$$

The sequence A_3 *is increasing and all the other sequences are unimodal.*

Proof. We have

$$\begin{aligned}
\tilde{\alpha}_{m,n} < \tilde{\alpha}_{m,n+2} &\iff \prod_{j=1}^{m} \frac{2(\gamma_{m,n} - j)}{n-j} < \prod_{j=1}^{m} \frac{2(\gamma_{m,n+2} - j)}{(n+2)-j} \\
&\iff (n+1)n(\gamma_{m,n} - m) < \gamma_{m,n}(n-m+1)(n-m) \\
&\iff \gamma_{m,n}(2n - m + 1) < n^2 + n
\end{aligned}$$

since $\gamma_{m,n+2} = \lceil (n+2)/2 + m/3\rceil = \gamma_{m,n} + 1$. We consider the various cases for $n \pmod 2$ and $m \pmod 3$.

Case 1: For $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 3$,

$$\begin{aligned}
n^2 + n - \gamma_{m,n}(2n - m + 1) &= n^2 + n - \left(\frac{n}{2} + \frac{m}{3}\right)(2n - m + 1) \\
&= \tfrac{1}{6}[(3-m)n + 2m(m-1)].
\end{aligned}$$

When $m = 3$, we see that $n^2 + n - \gamma_{m,n}(2n - m + 1) = 2 > 0$ and the sequence A_3 is always increasing in n.

For the other values $m = 3r$ with $r > 1$,

$$n^2 + n - \gamma_{m,n}(2n - m + 1) = \frac{(3-m)}{6}\left[n - \left(2m + 8 + \frac{24}{m-3}\right)\right] > 0$$

if and only if $n < 2m + 8 + 24/(m-3)$. The sequences A_{3r} with $r > 1$ are thus unimodal.

Case 2: For $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 3$,

$$\begin{aligned} n^2+n-\gamma_{m,n}(2n-m+1) &= n^2+n-\left(\frac{n}{2}+\frac{m}{3}+\frac{1}{6}\right)(2n-m+1) \\ &= \frac{(1-m)}{6}[n-(2m+1)] > 0 \end{aligned}$$

if and only if $n < 2m+1$. Since $n \geqslant 2m+1$ in this case, the sequences B_{3r+1} with $r \geqslant 1$ are all non-increasing.

Case 3: For the remaining cases, we have $\gamma_{m,n} \geqslant n/2+m/3+1/3$ and the inequality

$$n^2+n-\gamma_{m,n}(2n-m+1) < 0$$

holds for all $n \geqslant 2m$. In these cases, the sequences of both types are strictly decreasing. □

Lemma 3.4. *For $n \geqslant 2m$ we have $\tilde{\alpha}_{m,n} > 1$ except for $\tilde{\alpha}_{3,2t}$ with $t \geqslant 3$ and $\tilde{\alpha}_{6,12}$.*

Proof. Consider the terms $\tilde{\alpha}_{3,2t}$ with $t \geqslant 3$. In this case, $\gamma_{3,2t} = \lceil 2t/2+3/3 \rceil = t+1$ and

$$\tilde{\alpha}_{3,2t} = \frac{2^3(\gamma_{3,2t}-1)(\gamma_{3,2t}-2)(\gamma_{3,2t}-3)}{(2t-1)(2t-2)(2t-3)} = \frac{4t^2-8t}{4t^2-5t+3} < 1.$$

For the term $\tilde{\alpha}_{6,12}$ we have $\gamma_{6,12} = \lceil 12/2+6/3 \rceil = 8$ and $\tilde{\alpha}_{6,12} = 32/33 < 1$. Also, $\tilde{\alpha}_{6,13} = 64/33 > 1$ and $\tilde{\alpha}_{6,14} = 448/429 > 1$.

To show that the remaining terms are greater than 1, we adopt the following strategy. Recall from Lemma 3.3 that the following sequences

$$\begin{aligned} A_m &= \{\tilde{\alpha}_{m,2t} : t = m, m+1, \ldots\}, \\ B_m &= \{\tilde{\alpha}_{m,2t+1} : t = m, m+1, \ldots\} \end{aligned}$$

are unimodal or else decreasing with the exception of A_3 which is increasing. Indeed the proof of Lemma 3.3 shows that except for A_3, all the other sequences are strictly decreasing for sufficiently large t. We show that all these sequences approach the limit 1 as t tends to infinity. If the first term of such a sequence is greater than 1, then all the terms in that sequence will be greater than 1 too due to the unimodal property. With the exception of A_3 and A_6, we will show that all the other sequences have their first term bigger than 1. In the case of A_6, only the first term $\tilde{\alpha}_{6,12} = 32/33$ is less than 1, and the remaining terms are greater than 1 since $\tilde{\alpha}_{6,14} = 448/429 > 1$.

We show that all the sequences have limit equal to 1. For each $\lfloor n/2 \rfloor \geqslant m \geqslant 2$ and $j = 1, \ldots, m$, we have

$$\lim_{n\to\infty} \frac{2(\gamma_{m,n}-j)}{(n-j)} = \lim_{n\to\infty} \frac{2\left(\lceil n/2+m/3 \rceil - j\right)}{n-j} = 1.$$

Therefore,

$$\lim_{n\to\infty} \tilde{\alpha}_{m,n} = \lim_{n\to\infty} \prod_{j=1}^{m} \frac{2(\gamma_{m,n} - j)}{n - j} = 1.$$

It remains to show that the terms of the form $\tilde{\alpha}_{m,2m}$ and $\tilde{\alpha}_{m,2m+1}$ are all bigger than 1 except for $\tilde{\alpha}_{3,6}$ and $\tilde{\alpha}_{6,12}$. We divide these terms according to the value of m taken modulo 3. We look at the condition for these terms to be increasing with respect to m. We note that

$$\gamma_{m+3,2(m+3)} = \left\lceil \frac{2(m+3)}{2} + \frac{m+3}{3} \right\rceil = \left\lceil \frac{2m}{2} + \frac{m}{3} \right\rceil + 4 = \gamma_{m,2m} + 4.$$

We have

$$\begin{aligned} &\tilde{\alpha}_{m,2m} < \tilde{\alpha}_{m+3,2(m+3)} \\ &\quad \Longleftrightarrow \prod_{j=1}^{m} \frac{2\left(\gamma_{m,2m} - j\right)}{2m - j} < \prod_{j=1}^{m+3} \frac{2\left(\gamma_{m+3,2(m+3)} - j\right)}{2(m+3) - j} \\ &\quad \Longleftrightarrow (\gamma_{m,2m} - m)\prod_{i=0}^{2}(2m + 2i + 1) < \prod_{i=0}^{3}(\gamma_{m,2m} + i). \qquad (*) \end{aligned}$$

Similarly for the terms of the form $\tilde{\alpha}_{m,2m+1}$, we have

$$\begin{aligned} &\tilde{\alpha}_{m,2m+1} < \tilde{\alpha}_{m+3,2(m+3)+1} \\ &\quad \Longleftrightarrow \prod_{j=1}^{m} \frac{2\left(\gamma_{m,2m+1} - j\right)}{2m + 1 - j} < \prod_{j=1}^{m+3} \frac{2\left(\gamma_{m+3,2(m+3)+1} - j\right)}{2(m+3) + 1 - j} \\ &\quad \Longleftrightarrow (\gamma_{m,2m+1} - m)\prod_{i=0}^{2}(2m + 2i + 1) < \prod_{i=0}^{3}(\gamma_{m,2m+1} + i). \qquad (**) \end{aligned}$$

Let

$$D_m(*) = \prod_{i=0}^{3}(\gamma_{m,2m} + i) - (\gamma_{m,2m} - m)\prod_{i=0}^{2}(2m + 2i + 1)$$

and

$$D_m(**) = \prod_{i=0}^{3}(\gamma_{m,2m+1} + i) - (\gamma_{m,2m+1} - m)\prod_{i=0}^{2}(2m + 2i + 1).$$

These are differences obtained by subtracting the left-hand side from the right-hand side of the inequalities in $(*)$ and $(**)$, respectively.

Case 1: $m = 3r$ where $r \geqslant 2$. Here $\gamma_{m,2m} = \lceil 2m/2 + m/3 \rceil = 4r$ and $\gamma_{2m,2m+1} = \lceil (2m+1)/2 + m/3 \rceil = 4r + 1$.

Using these values we have

$$D_m(*) = r(2r + 1)(20r^2 + 20r + 9) > 0,$$

$$D_m(**) = (r + 1)(2r + 1)(20r^2 + 20r + 9) > 0.$$

It follows that

$$1 < \frac{256}{221} = \tilde{\alpha}_{9,18} < \tilde{\alpha}_{12,24} < \cdots < \tilde{\alpha}_{3r,6r} < \tilde{\alpha}_{3(r+1),6(r+1)} < \cdots$$

and

$$1 < \frac{64}{33} = \tilde{\alpha}_{6,13} < \tilde{\alpha}_{9,19} < \cdots < \tilde{\alpha}_{3r,6r+1} < \tilde{\alpha}_{3(r+1),6(r+1)+1} < \cdots$$

We note that in the case of A_6, $\tilde{\alpha}_{6,12} = 32/33 < 1$ and $\tilde{\alpha}_{6,14} = 448/429 > 1$.

Case 2: $m = 3r + 1$ where $r \geqslant 1$. Here

$$\gamma_{m,2m} = \lceil 2m/2 + m/3 \rceil = 4r + 2$$

and

$$\gamma_{2m,2m+1} = \lceil (2m + 1)/2 + m/3 \rceil = 4r + 2.$$

Using these values we have

$$D_m(*) = D_m(**) = (r + 1)(2r + 1)(20r^2 + 40r + 15) > 0.$$

It follows that

$$1 < \frac{16}{7} = \tilde{\alpha}_{4,8} < \tilde{\alpha}_{7,14} < \cdots < \tilde{\alpha}_{3r+1,6r+2} < \tilde{\alpha}_{3(r+1)+1,6(r+1)+2} < \cdots$$

and

$$1 < \frac{8}{9} = \tilde{\alpha}_{4,9} < \tilde{\alpha}_{7,15} < \cdots < \tilde{\alpha}_{3r+1,6r+3} < \tilde{\alpha}_{3(r+1),6(r+1)+1} < \cdots$$

Case 3: $m = 3r + 2$ where $r \geqslant 0$. Here

$$\gamma_{m,2m} = \lceil 2m/2 + m/3 \rceil = 4r + 3$$

and

$$\gamma_{2m,2m+1} = \lceil (2m + 1)/2 + m/3 \rceil = 4r + 4.$$

Using these values we have

$$D_m(*) = (r + 1)(2r + 3)(20r^2 + 40r + 15) > 0,$$
$$D_m(**) = (2r + 3)(20r^3 + 70r^2 + 127r + 160) > 0.$$

It follows that

$$1 < \frac{16}{7} = \tilde{\alpha}_{2,4} < \tilde{\alpha}_{5,10} < \cdots < \tilde{\alpha}_{3r+2,6r+4} < \tilde{\alpha}_{3(r+1)+2,6(r+1)+4} < \cdots$$

and

$$1 < 2 = \tilde{\alpha}_{2,5} < \tilde{\alpha}_{5,11} < \cdots < \tilde{\alpha}_{3r+2,2(3r+2)+1} < \tilde{\alpha}_{3(r+1)+2,6(r+1)+5} < \cdots$$

Thus all the first terms except for $\tilde{\alpha}_{3,6}$ and $\tilde{\alpha}_{6,12}$ are greater than 1. This completes the proof. □

Remark. In view of the final equation in (5) and Corollary 3.2, we see that for a tree on n vertices with a maximum matching of size m we have $\lceil n/2 + m/3 \rceil \geqslant \kappa(T)$

with some exceptions. The exceptions are trees on an even number of vertices with a maximum matching of size 3, and those on 12 vertices with a maximum matching of size 6. We shall analyze the exceptions in the following section.

4. Handling the exceptions

We note that the terms $\tilde{\alpha}_{3,2t}$ with $t \geqslant 3$ are less than 1. We have to show that all trees T on $2t$ vertices having a maximum matching of size 3 satisfy $\kappa(T) \leqslant \lceil 2t/2 + 3/3 \rceil = t + 1$. We note that $\gamma_{3,2t} = \lceil 2t/2 + 3/3 \rceil = t + 1$. Let $a(1)$, $a(2)$ and $a(3)$ denote the 1-, 2- and 3-vertex orientations in the tree, respectively. We recall that $a(1) = 2t - 1$.

$$\begin{aligned}
\bar{d}_{t+1}(T) &= \frac{2(\gamma_{3,2t} - 1)a(1)}{(2t - 1)} + \frac{4(\gamma_{3,2t} - 1)(\gamma_{3,2t} - 2)a(2)}{(2t - 1)(2t - 2)} \\
&\quad + \frac{8(\gamma_{3,2t} - 1)(\gamma_{3,t} - 2)(\gamma_{3,2t} - 3)a(3)}{(2t - 1)(2t - 2)(2t - 3)} \\
&= \frac{n \times (n - 1)}{(n - 1)} + \frac{n \times a(2)}{(n - 1)} + \frac{n(n - 4) \times a(3)}{(n - 1)(n - 3)} \\
&= n + a(2) + \frac{a(2)}{(n - 1)} + a(3) - \frac{3 \times a(3)}{(n - 1)(n - 3)} \\
&= a(1) + a(2) + a(3) + 1 + \frac{(n - 3) \times a(2) - 3 \times a(3)}{(n - 1)(n - 3)} \\
&= h(T) + 1 + \frac{E}{(n - 1)(n - 3)},
\end{aligned}$$

where $E = (n - 3) \times a(2) - 3 \times a(3)$.

In order to show that $\bar{d}_k(L(T)) > h(T)$, we will show that $E > 0$ for trees with 3 matchings. The alphabets a, b and c in Fig. 2 indicate the number of leaves that are attached at those positions.

Type 1:

$$\begin{aligned}
&n = a + b + 4, \\
&a(1) = a + b + 3, \\
&a(2) = 3a + 3b + 3ab + 1, \\
&a(3) = ab, \\
&E = 3a^2 + 6ab + 3a^2b + 4a + 3b^2 + 3ab^2 + 4b + 1 > 0,
\end{aligned}$$

Type 2:

$$\begin{aligned}
&n = a + b + 5, \\
&a(1) = a + b + 4, \\
&a(2) = 6a + 6b + 4ab + 4,
\end{aligned}$$

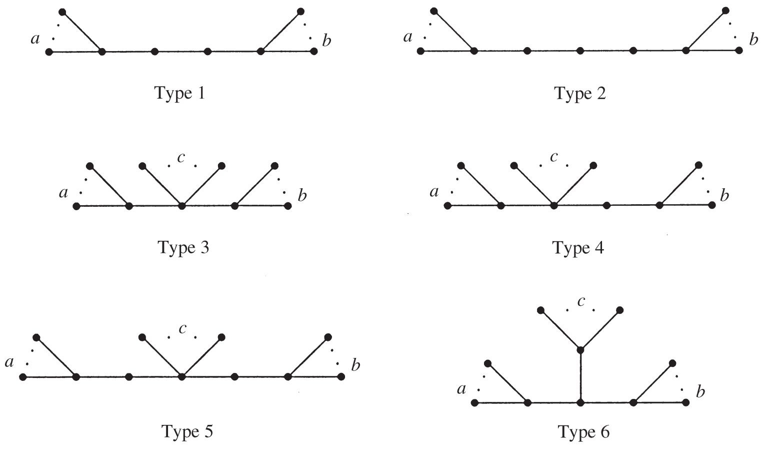

Fig. 2.

$$a(3) = a + b + 4ab,$$
$$E = 6a^2 + 8ab + 4a^2b + 13a + 6b^2 + 4ab^2 + 13b + 8 > 0.$$

Type 3:

$$n = a + b + c + 3,$$
$$a(1) = a + b + c + 2,$$
$$a(2) = a + b + ac + bc + 2ab,$$
$$a(3) = abc,$$
$$E = a^2 + 2ab + a^2c + abc + 2a^2b + b^2 + b^2c + 2ab^2 + ac + bc + ac^2 + bc^2 > 0.$$

Type 4:

$$n = a + b + c + 4,$$
$$a(1) = a + b + c + 3,$$
$$a(2) = 3a + 3b + c + ac + 2bc + 3ab,$$
$$a(3) = ac + ab + 2abc,$$
$$E = 3a^2 + 6ab + 2ac + a^2c + 3a^2b + 3b^2 + 6bc + 2b^2c + 3ab^2 + c^2 + ac^2 + 2bc^2 + 3a + 3b + c > 0.$$

Type 5:

$$n = a + b + 5,$$
$$a(1) = a + b + c + 4,$$

$$a(2) = 6a + 6b + 2c + 2ac + 2bc + 4ab + 4,$$
$$a(3) = a + b + c + 2ac + 2bc + 4ab + 4abc,$$
$$E = 8 + 13a + 13b + 2c^2 + 8ab + 6a^2 + 4a^2b + 6b^2 + 4ab^2 + 6ac + 5c$$
$$- 4abc + 6bc + 2a^2c + 2b^2c + 2ac^2 + 2bc^2 > 0.$$

This is because either $4a^2b > 4abc$, $4ab^2 > 4abc$ or $2ac^2 + 2bc^2 > 4abc$ is true.

Type 6:

$$n = a + b + c + 4,$$
$$a(1) = a + b + c + 3,$$
$$a(2) = 2a + 2b + 2c + 2ac + 2bc + 2ab,$$
$$a(3) = ac + bc + ab + 3abc,$$
$$E = 2a^2 + 3ab + 3ac + 2a^2c - 3abc + 2a^2b + 2b^2 + 3bc$$
$$+ 2b^2c + 2ab^2 + 2c^2 + 2ac^2 + 2bc^2 + 2a + 2b + 2c > 0,$$

where without loss of generality, we may assume $a \geqslant b \geqslant c$. Thus, $2a^2b + 2a^2c > 3abc$.

For the case when $m = 6$ and $n = 12$, we can calculate all the coefficients needed. Moreover, $a(1) = 11$ and $a(6) = 1$. Thus we show that $\bar{d}_{\lceil n/2+m/3\rceil}(T) > h(T)$ as follows:

$$\begin{aligned}\bar{d}_{\lceil n/2+m/3\rceil}(T) &= \frac{14}{11}a(1) + \frac{84}{55}a(2) + \frac{56}{33}a(3) + \frac{56}{33}a(4) + \frac{16}{11}a(5) + \frac{32}{33}a(6) \\ &> 14 + a(2) + a(3) + a(4) + a(5) \\ &= a(1) + a(2) + a(3) + a(4) + a(5) + a(6) + 2 > h(T).\end{aligned}$$

Theorem 4.1. *For a tree T with n vertices and a maximum matching of size m, we have* $\bar{d}_{\lceil n/2+m/3\rceil}(T) \geqslant h(T)$. *It follows that* $\lceil n/2 + m/3\rceil \geqslant \kappa(T)$.

Proof. This claim follows from Lemmas 3.1, 3.3 and 3.4 and the results in this section. □

5. Lower bound of $\kappa_1(A)$

In the earlier sections, we found the upper bound, $\lceil n/2 + m/3\rceil$, of $\kappa(T)$ by ensuring mainly that $\alpha(n, k;\ j) > 1$ for all j in (5). This approach may seem too generous in the sense that $\bar{d}_k(T) - h(T)$ may be positive without requiring all the $\alpha(n, k;\ j)$'s to be greater than 1. In this section, we show that we cannot reduce $\kappa(T)$ to $\lceil n/2 + m/4\rceil$ since there exist trees T with $\bar{d}_{\lceil n/2+m/4\rceil}(T) < h(T)$ for arbitrarily large *n* and *m*. This bounds the coefficient κ_1 in (4) to the range $1/3 \geqslant \kappa_1 > 1/4$.

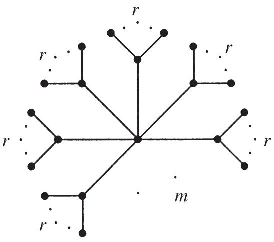

Fig. 3. Tree T_r.

Consider the following tree T_r with a central vertex and m branches, each of which has r leaves as shown in Fig. 3. The number of vertices in T_r is $n = mr + m + 1$.

Let m be fixed with $m \equiv 2 \pmod 4$, and it is clear that n is odd since m is even. We wish to take r to be large. Let

$$\gamma' = \left\lceil \frac{n}{2} + \frac{m}{4} \right\rceil = \frac{n}{2} + \frac{m}{4} = \frac{mr}{2} + \frac{3m}{4} + \frac{1}{2}.$$

To compute the number of j-vertex orientations, $a_{T_r}(j)$, we divide the j-vertex orientations into those in which the central vertex is adjacent to an edge with two arrows on it, and the rest. The former can be counted as $m \times \binom{m-1}{j-1} \times r^{j-1}$ and the number of j-vertex orientations of the other type is given by $j \times \binom{m}{j} \times r^j$. Hence

$$\begin{aligned}
a_{T_r}(j) &= \text{total number of } j\text{-vertex orientations} \\
&= \text{number of } j\text{-vertex orientations where the central vertex} \\
&\quad\ \text{is not adjacent to an edge with two arrows} \\
&\quad + \text{number of } j\text{-vertex orientations where the central vertex} \\
&\quad\ \text{is adjacent to an edge with two arrows on it} \\
&= j \times \binom{m}{j} \times r^j + m \times \binom{m-1}{j-1} \times r^{j-1} \\
&= j \times \binom{m}{j} \times (r^j + r^{j-1}).
\end{aligned}$$

It is clear that the tree T_r has a maximum matching of size m. For $j = 1, \ldots, m$, when r is sufficiently large we can write

$$\begin{aligned}
\alpha(n, \gamma'; j) &= \frac{2^j(\gamma' - 1)(\gamma' - 2) \cdots (\gamma' - j)}{(n-1)(n-2) \cdots (n-j)} \\
&= \prod_{i=1}^{j} \frac{2(n/2 + m/4 - i)}{(n-i)} \\
&= \prod_{i=1}^{j} \frac{(mr + m + 1 + m/2 - 2i)}{(mr + m + 1 - i)}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{j} \frac{mr[1+(3/2+(1-2i)/m)1/r]}{mr[1+(1+(1-i)/m)1/r]} \\
&= \prod_{i=1}^{j} \left[1+\left(\frac{3}{2}+\frac{(1-2i)}{m}\right)\frac{1}{r}\right]\left[1-\left(1+\frac{1-i}{m}\right)\frac{1}{r}+\mathrm{O}\left(\frac{1}{r^2}\right)\right] \\
&= \prod_{i=1}^{j} \left[1+\left(\left(\frac{3}{2}+\frac{1-2i}{m}\right)-\left(1+\frac{1-i}{m}\right)\right)\frac{1}{r}+\mathrm{O}\left(\frac{1}{r^2}\right)\right] \\
&= \prod_{i=1}^{j} \left[1+\left(\frac{1}{2}-\frac{i}{m}\right)\frac{1}{r}+\mathrm{O}\left(\frac{1}{r^2}\right)\right] \\
&= 1+\frac{1}{r}\sum_{i=1}^{j}\left(\frac{1}{2}-\frac{i}{m}\right)+\mathrm{O}\left(\frac{1}{r^2}\right) \\
&= 1+\left(\frac{j}{2}-\frac{j(j+1)}{2m}\right)\frac{1}{r}+\mathrm{O}\left(\frac{1}{r^2}\right) \\
&= 1+\frac{j(m-j-1)}{2mr}+\mathrm{O}\left(\frac{1}{r^2}\right).
\end{aligned}
$$

In particular, we have the following:

$$
\begin{aligned}
\alpha(n,\gamma';m) &= 1-\frac{1}{2r}+\mathrm{O}\left(\frac{1}{r^2}\right) \\
\alpha(n,\gamma';m-1) &= 1+\mathrm{O}\left(\frac{1}{r^2}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\bar{d}_{\lceil n/2+m/4\rceil}(T)-h(T) &= \sum_{j=1}^{m} a_T(j)(\alpha(n,\gamma';j)-1) \\
&= \sum_{j=1}^{m}\left[j\times\binom{m}{j}\times(r^j+r^{j-1})\right]\left[\frac{j(m-j-1)}{2mr}+\mathrm{O}\left(\frac{1}{r^2}\right)\right] \\
&= \left[-\frac{m}{2}\times r^{m-1}+\mathrm{O}(r^{m-2})\right]+\left[\mathrm{O}(r^{m-3})\right]+\sum_{j=1}^{m-2}\mathrm{O}\left(r^{j-1}\right) \\
&= -\frac{mr^{m-1}}{2}+\mathrm{O}\left(r^{m-2}\right) \\
&< 0 \quad \text{for sufficiently large } r.
\end{aligned}
$$

There exists a sufficiently large r such that $\bar{d}_{\lceil n/2+m/4\rceil}(T_r) < h(T_r)$, where T_r is the tree described in Fig. 3. We note that n and m can be taken to be arbitrarily large.

In the notation of (4) we have the bounds on κ_1 as $1/3 \geqslant \kappa_1 > 1/4$. We have not yet obtained bounds on κ_2. This will be a natural extension which can help us understand the relative position of $h(T)$ in the sequence of normalized hook immanants.

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